can be extended to two-dimensional initial-boundary value UGT problems on replacing the fundamental solutions.

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A PROBLEM IN ELASTICITY THEORY*

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The problem of determining the dimensions of the transverse cross-sections of a beam from the given frequencies of its natural vibrations is examined. Frequency spectra are indicated that determine the dimenions of the transverse cross-sections of the beam uniquely, an effective procedure is presented for solving the inverse problem, and a uniqueness theorem is proved. The method of standard models /1/ is used to solve the inverse problem.

We examine the differential equation describing beam vibrations in the form

$$(h^{\mu}(x)\,y'')'' = \lambda h(x)\,y, \quad 0 \leqslant x \leqslant T \tag{1}$$

here $h(x)$ is a function characterizing the beam transverse section, and $\mu = 1, 2, 3$ is a fixed number. We will assume that the function $h(x)$ is absolutely continuous in the segment $[0, T]$ and $h(x) > 0$, $h(0) = 1$. The inverse problem for (1) in the case $\mu = 2$ (similar transverse sections) was investigated /2/ in determining small changes in the beam transverse sections for given small changes in a finite number of its natural vibration frequencies.

Let $\{\lambda_{kj}\}_{k\geqslant 1, j=1,2}$ be the eigenvalues of boundary-value problems Q_j for (1) with the boundary conditions

$$y(0) = y^{(j)}(0) = y(T) = y'(T) = 0$$

The inverse problem is formulated as follows.

Problem 1. Find the function $h(x)$, $x \in [0, T]$ for given frequency spectra $\{\lambda_{kj}\}_{k\geqslant 1, j=1,2}$.

To solve this inverse problem we will first prove several auxiliary assertions.

We consider the function $\Phi(x, \lambda)$ the solution of (1) under the conditions $\Phi(0, \lambda) = \Phi(T, \lambda) = \Phi'(T, \lambda) = 0$, $\Phi'(0, \lambda) = 1$. We set $\alpha(\lambda) = \Phi''(0, \lambda)$. Furthermore, let the functions $C_{\nu}(x, \lambda)$ $(\nu = 0, 1, 2, 3)$ be solutions of (1) under the initial conditions $C_{\nu}^{(\mu)}(0, \lambda) = \delta_{\nu\mu}$, $\nu, \mu = 0, 1, 2, 3$. We will use the notation $\Delta_j(\lambda) = C_{3-j}(T, \lambda)\,C_3'(T, \lambda) - C_3(T, \lambda)\,C_{3-j}'(T, \lambda)$, $j = 1, 2$

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$$\gamma(x) = \int_0^x (h(t))^{(1-\mu)/4} dt, \quad \tau = \gamma(T)$$

It is obvious that

$$\Phi(x, \lambda) = \det [C_\nu(x, \lambda), C_\nu(T, \lambda), C_\nu'(T, \lambda)]_{\nu=1,2,3} / \Delta_1(\lambda)$$

and therefore

$$\alpha(\lambda) = -\Delta_2(\lambda)/\Delta_1(\lambda) \tag{2}$$

Let $\lambda = \rho^4$, $S = \{\rho: \arg\rho \in (0, \pi/4)\}$. It is known (see /3, 4/, say) that the following asymptotic formulas hold

$$\lambda_{kj} = (k\pi\tau^{-1})^4 (1 + A_{j1}k^{-1} + O(k^{-2})), \quad k \to \infty \tag{3}$$

$$\Delta_j(\lambda) = \rho^{j-5} A_{j2} \exp(\rho(1 - i)\tau)(1 + O(\rho^{-1})) \tag{4}$$

$$\Delta_j(\lambda) = O(\rho^{j-5} \exp(C|\lambda|^{1/4})) \tag{5}$$

$$\Phi^{(\nu)}(x, \lambda) = \rho^{\nu-1} \sum_{\xi=1}^{2} (R_\xi \gamma'(x))^\nu g_\xi(x) \exp(\rho R_\xi \gamma(x))(1 + O(\rho^{-1}));$$
$$R_1 = -1, \quad R_2 = i \tag{6}$$
$$\alpha(\lambda) = \rho(1 - i)(1 + O(\rho^{-1}))$$

as $|\lambda| \to \infty$, $\rho \in S$ where the numbers $A_{j\nu}$ depend on τ and the functions $g_\xi(x)$ are absolutely continuous $g_\xi(x) > 0$, $g_1(0) = -g_2(0) = (-1 - i)^{-1}$.

Lemma 1. The function $\alpha(\lambda)$ is defined uniquely by giving the spectra $\{\lambda_{kj}\}_{k\geqslant 1, j=1,2}$.

Proof. The eigenvalues $\{\lambda_{kj}\}$ of the boundary-value problems Q_j are identical with the zeros of the entire functions $\Delta_j(\lambda)$ analytic in λ. Indeed, let λ^* be an eigenvalue and $\psi(x)$ an eigenfunction of the boundary-value problem Q_j. Then

$$\psi(x) = \sum_{\mu=0}^{3} \beta_\mu C_\mu(x, \lambda^*)$$

where

$$\sum_{\mu=0}^{3} \beta_\mu C_\mu^{(k)}(0, \lambda^*) = 0, \ \sum_{\mu=0}^{3} \beta_\mu C_\mu^{(s)}(T, \lambda^*) = 0; \quad k = 0, j; \quad s = 0, 1$$

Since $\psi(x) \not\equiv 0$ this linear homogeneous algebraic system has non-zero solutions and, therefore, its determinant equals zero, i.e., $\Delta_j(\lambda^*) = 0$. Repeating all the reasoning in reverse order, we obtain that if $\Delta_j(\lambda^*) = 0$ then λ^* is an eigenvalue of the boundary-value problem Q_j.

It follows from (5) that the order of the functions $\Delta_j(\lambda)$ equals $1/4$ and, therefore, according to Borel's theorem /5/

$$\Delta_j(\lambda) = B_j \Pi (1 - \lambda/\lambda_{kj}), \ B_j = \text{const} \tag{7}$$

Here and everywhere later, the product is evaluated over $k = 1, 2, \ldots$

Let us examine the positive function $h^\circ(x)$, $h^\circ(0) = 1$ that is absolutely continuous in the segment $[0, T]$. We will agree that if a certain symbol p denotes an object referring to (1) and constructed according to the function $h(x)$, then p° is an analogous object constructed according to the function $h^\circ(x)$.

Let $\tau^\circ = \tau$. We have from (7)

$$\frac{\Delta_j(\lambda)}{\Delta_j^\circ(\lambda)} = \frac{B_j S_{j1}(\lambda)}{B_j^\circ S_j}, \quad S_j = \Pi \frac{\lambda_{kj}}{\lambda_{kj}^\circ}, \quad S_{j1}(\lambda) = \Pi\left(1 - \frac{\lambda_{kj}^\circ - \lambda_{kj}}{\lambda_{kj}^\circ - \lambda}\right)$$

By virtue of Eqs.(3) and (4) $\lim \Delta_j(\lambda)/\Delta_j^\circ(\lambda) = 1$, $\lim S_{j1}(\lambda) = 1$ as $|\lambda| \to \infty$, $\rho \in S$ and, therefore

$$B_j = B_j^\circ S_j \tag{8}$$

We obtain from (2) and (7)

$$\alpha(\lambda) = B\Pi \frac{\lambda_{k1}}{\lambda_{k2}} \frac{\lambda_{k2} - \lambda}{\lambda_{k1} - \lambda}, \quad B = -\frac{B_2}{B_1}$$

or, taking account of (8),

$$\alpha(\lambda) = B^\circ \Pi \frac{\lambda_{k1}^\circ}{\lambda_{k2}^\circ} \frac{\lambda_{k2} - \lambda}{\lambda_{k1} - \lambda}$$

Hence, the assertion of Lemma 1 follows.

Lemma 2. Let $p(x) = h^{\mu}(x)$. The following relationship holds:

$$\int_0^T ((h(x) - h^{\circ}(x))\lambda\Phi(x,\lambda)\Phi^{\circ}(x,\lambda) - (p(x) - p^{\circ}(x))\times \Phi''(x,\lambda)\Phi^{\circ\prime\prime}(x,\lambda))\,dx = \alpha(\lambda) - \alpha^{\circ}(\lambda) \tag{9}$$

Proof. Let

$$l_{\lambda}y = (p(x)y'')'' - \lambda h(x)y$$
$$L(y,z) = (p(x)y'')'z - p(x)y''z' + p(x)y'z'' - y(p(x)z'')'$$

Then

$$\int_0^T l_{\lambda}y(x)z(x)\,dx = L(y(x), z(x))\Big|_0^T + \int_0^T y(x)\,l_{\lambda}z(x)\,dx \tag{10}$$

Using relationships (10), the equalities $l_{\lambda}\Phi(x,\lambda) = l_{\lambda}^{\circ}\Phi^{\circ}(x,\lambda) = 0$ and the boundary conditions on the functions $\Phi(x,\lambda)$, $\Phi^{\circ}(x,\lambda)$, we obtain

$$\int_0^T \Phi^{\circ}(x,\lambda)(l_{\lambda} - l_{\lambda}^{\circ})\Phi(x,\lambda)\,dx = -L^{\circ}(\Phi(x,\lambda), (\Phi^{\circ}(x,\lambda))\Big|_0^T - \int_0^T \Phi(x,\lambda)\,l_{\lambda}^{\circ}\Phi^{\circ}(x,\lambda)\,dx =$$
$$\Phi'(0,\lambda)\Phi^{\circ\prime\prime}(0,\lambda) - \Phi''(0,\lambda)\Phi^{\circ\prime}(0,\lambda) = \alpha^{\circ}(\lambda) - \alpha(\lambda)$$

On the other hand, integrating the left-hand side of the last equality by parts, we have

$$\int_0^T \Phi^{\circ}(x,\lambda)(l_{\lambda} - l_{\lambda}^{\circ})\Phi(x,\lambda)\,dx = (((p(x) - p^{\circ}(x))\Phi''(x,\lambda))'\Phi^{\circ}(x,\lambda) -$$
$$(p(x) - p^{\circ}(x))\Phi''(x,\lambda)\Phi^{\circ\prime}(x,\lambda)\Big|_0^T + \int_0^T ((p(x) - p^{\circ}(x))\Phi''(x,\lambda)\Phi^{\circ\prime\prime}(x,\lambda) -$$
$$\lambda(h(x) - h^{\circ}(x))\Phi(x,\lambda)\Phi^{\circ}(x,\lambda))\,dx$$

Since the substitution vanishes, we hence obtain relationship (9).

Lemma 3. Consider the integral

$$J(z) = \int_0^T f(x)H(x,z)\,dx \tag{11}$$
$$f(x) = (f_n + s(x))x^n/n!, \quad s(x) \in C[0,T], \quad s(0) = 0, \ n \geqslant 0$$
$$H(x,z) = e^{-za(x)}(1 + \xi(x,z)/z)$$
$$a(x) \in C^1[0,T], \ 0 < a(x_1) < a(x_2) \ (0 < x_1 < x_2)$$
$$a^{(\nu)}(x) \sim \beta x^{1-\nu} \quad (x \to +0, \ \nu = 0, 1), \quad a'(x) > 0$$

where the function $\xi(x,z)$ is continuous and bounded for $x \in [0,T]$, $z \in G \doteq \{z\colon \arg z \in [-\pi/2 + \delta_0, \pi/2 - \delta_0], \ \delta_0 > 0\}$. Then as $|z| \to \infty$, $z \in G$

$$J(z) = (\beta z)^{-n-1}(f_n + o(1))$$

Proof. Case 1. Let $a(x) \equiv x$. Then

$$z^{n+1}J(z) = f_n z^{n+1}\int_0^T E(x,z)\,dx + z^{n+1}\int_0^T s(x)E(x,z)\,dx + z^n\int_2^T f(x)e^{-zx}\xi(x,z)\,dx =$$
$$J_1(z) + J_2(z) + J_3(z), \ E(x,z) = e^{-zx}x^n/n!$$

The estimate $\operatorname{Re} z \geqslant \varepsilon_0|z|$, $\varepsilon_0 > 0$ holds in the domain G. Since

$$\int_0^{\infty} E(x,z)\,dx = z^{-n-1}$$

then

$$J_1(z) = f_n - f_n z^{n+1}\int_T^{\infty} E(x,z)\,dx$$

and therefore $J_1(z) - f_n \to 0$ as $|z| \to \infty,\ z \in G$.

Let $\varepsilon > 0$. We select $\delta = \delta(\varepsilon)$ such that $|s(x)| < \varepsilon_0^{n+1}\varepsilon/2$ for $x \in [0, \delta]$. Then

$$|J_2(z)| < \varepsilon/2\,(\varepsilon_0|z|)^{n+1}\int_0^{\delta} E(x, -\varepsilon_0|z|)\,dx + |z|^{n+1}\int_{\delta}^{T} |s(x)|\,E(x, -\varepsilon_0|z|)\,dx < \varepsilon/2 +$$

$$|z|^{n+1}e^{-\varepsilon_0|z|\delta}\int_0^{T-\delta} |s(x+\delta)|\,e^{-\varepsilon_0|z|x}(x+\delta)^n/n!\,dx$$

As $|z| \to \infty,\ z \in G$ the second component can be made less than $\varepsilon/2$. By virtue of the arbitrariness of ε we have $J_2(z) \to 0$ as $|z| \to \infty,\ z \in G$.

Since $|f_n + s(x)|\,|\xi(x, z)| < C$, then for $z \in G$

$$|J_3(z) < C|z|^n\int_0^T E(x, -\varepsilon_0|z|)\,dx < C|z|^{-1}\varepsilon_0^{-n-1}$$

i.e., $J_3(z) \to 0$ as $|z| \to \infty,\ z \in G$. Therefore the lemma is proved in Case 1.

Case 2. Now let $a(x)$ be an arbitrary function satisfying the conditions of the lemma. Then the function $t = a(x)$ has the inverse $x = b(t)$, where $b(t) \in C^1[0, T_1]$ where $T_1 = a(T)$; $b(t) > 0$ for $t > 0$ and $b^{(\nu)}(t) = \beta^{-1}t^{1-\nu}(1 + \theta_\nu(t))$, $\theta_\nu(t) \in C[0, T_1]$, $\theta_\nu(0) = 0$, $\nu = 0, 1$. Let us make the change of variable $t = a(x)$ in the integral in (11). We obtain

$$J(z) = \int_0^{T_1} f^*(t)\,H^*(t, z)\,dt$$

$$H^*(t, z) = e^{-zt}(1 + \xi(b(t), z)/z),\quad f^*(t) = b'(t)\,f(b(t))$$

It is clear that

$$f^*(t) = \frac{t^n}{\beta^{n+1}n!}(f_n + s^*(t)),\quad s^*(t) \in C[0, T_1],\quad s^*(0) = 0$$

Therefore, the problem reduces to Case 1 and Lemma 3 is proved.

Let us put

$$A_n = \frac{1}{(R_1 - R_2)^2}\sum_{k,j=1}^{2}\frac{(-1)^{k+j}(1 - \mu R_k^2 R_j^2)}{(R_k + R_j)^{n+1}},\quad n \geqslant 1$$

Since $R_1 = -1$ and $R_2 = i$, we calculate

$$A_n = \frac{a_n}{2i(-2)^{n+1}},\quad a_n = (\mu - 1)(1 + i^{n+1}) + 2(\mu + 1)(1 + i)^{n+1}$$

Taking account of the relationships $|1 + i^{n+1}| \leqslant \sqrt{2}$, $|1 + i|^{n-1} = (\sqrt{2})^{n+1}$ we obtain that $a_n \neq 0$, $n \geqslant 1$ and, therefore, $A_n \neq 0$ for all $n \geqslant 1$.

Lemma 4. As $x \to +0$ let

$$h(x) - h^\circ(x) \sim H_n x^n/n!$$

Then as $|\rho| \to \infty,\ \rho \in S$ there exists a finite limit

$$F_n = \lim \rho^{n-1}(\alpha(\lambda) - \alpha^\circ(\lambda))$$

where

$$A_n H_n = F_n \tag{12}$$

Proof. Since $p(x) = h^\mu(x)$ then by virtue of the conditions of the lemma we have as $x \to +0$

$$p(x) - p^\circ(x) \sim \mu H_n x^n/n!$$

Using the asytmptotic formulas (6) and Lemma 3 we find as $|\rho| \to \infty,\ \rho \in S$

$$\int_0^T (h(x) - h^\circ(x))\,\lambda\Phi(x, \lambda)\,\Phi^\circ(x, \lambda)\,dx \sim \frac{H_n}{2i\rho^{n-1}}\sum_{k,j=1}^{2}\frac{(-1)^{k+j}}{(R_k + R_j)^{n+1}}$$

$$\int_0^T (p(x) - p^\circ(x))\,\Phi''(x, \lambda)\,\Phi^{\circ''}(x, \lambda)\,dx \sim \frac{\mu H_n}{2i\rho^{n-1}}\sum_{k,j=1}^{2}\frac{(-1)^{k+j}R_k^2 R_j^2}{(R_k + R_j)^{n+1}}$$

Substituting the expressions obtained in (9), we obtain the assertion of Lemma 4.

Let A be a set of functions analytic in the segment $[0, T]$. The following results from the facts presented above

Theorem. Problem 1 has a unique solution in the class of functions $h(x) \in A$ where it can be found according to the following algorithm:

1) we construct the function $\alpha(\lambda)$ according to the given spectra $\{\lambda_{kj}\}_{k\geqslant 1,\, j=1,2}$

2) we calculate $h_n = h^{(n)}(0)$, $n \geqslant 0$, $h_0 = 1$; for this we successively perform operations for $n = 1, 2, \ldots$: we construct the function $h^\circ(x) \in A$, $h^\circ(x) > 0$ such that $h^{\circ(\nu)}(0) = h_\nu$, $\nu = 0, 1, \ldots, n-1$ and arbitrarily in the rest, and we calculate h_n from relationship (12), where $H_n = h_n - h_n^\circ$;

3) we determine the function $h(x)$ from the formula

$$h(x) = \sum_{n=0}^{\infty} h_n \frac{x^n}{n!}, \quad 0 < x < R, \quad R = \left[\overline{\lim_{n\to\infty}} \left(\frac{|h_n|}{n!}\right)^{1/n}\right]^{-1}$$

If $R < T$, then for $R < x < T$ the function $h(x)$ is constructed by analytic continuation.

We note that the inverse problem in the class of piecewise-analytic functions can also be solved in an analogous manner.

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